XVII. *On the Enumeration of* x-*edra having Triedral Summits, and an* (x−1)-*gonal Base. By the Rev.* Thomas P. Kirkman, *M.A. Communicated by* A. Cayley, *F.R.S.*

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It is easily proved that no x-edron has a face of more than $x-1$ angles, and that, if it has an $(x-1)$-gonal face, it has at least two triangular faces. The object of this paper is to determine the number of x-edra which have an $(x-1)$-gonal face, and all their summits triedral, or, which is the same thing, the number of x-acra which have an $(x-1)$-edral summit, and all their faces triangular.

We may call the $(x-1)$-gonal face the base of the x-edron. All the faces will be collateral with that base, and k of them will be triangular faces. If we suppose those k triangles to become infinitely small, in any x-edron A, we have as the result an $(x-k)$-edron B, having only triedral summits, none of whose triangular faces was a triangle of A. And it is evident that there is only one $(x-k)$-edron B from which A can be cut by sections that shall remove no edge entirely, and shall leave untouched no triangle of B. It is plain also that B cannot have more, but may have fewer, triangles than A; for if the vanishing of a triangle of A gave rise to two triangles in B, B, having two contiguous triangles, and all its summits triedral, would be a tetraedron.

If now we suppose the triangular faces of B to vanish, of which there must at least be two, there will arise a polyedron C, having only triedral summits, and fewer faces than B. In like manner C is reduced by the vanishing of its triangles to a solid of still fewer faces, and, by this continual evanescence of triangular faces, we shall finally arrive, either at a tetraedron, or a pentaedron with two triangular faces.

Hence it appears, that every x-edron having its base $(x-1)$-gonal, and all its summits triedral, can be cut from one of these two simple solids, by reversing the above process, *i. e.* by cutting away k summits of the base of a polyedron B, having $k-h$ triangular faces, so as to leave none of those $k-h$ triangles untouched. And by this process no polyedron A can be twice generated. It is to be remembered that we are all through handling no polyedra but those whose summits are all triedral.

The pentaedron on a 4-lateral base, has around that base the faces 3434. This I call a *doubly reversible* polyedron, as it exhibits in the faces about the base the series 34 repeated, and reads backwards and forwards the same.

If we cut every base summit of the tetraedron, we obtain a heptaedron having around its hexagonal base the faces 353535. This I call a *trebly reversible* heptaedron.

If we cut a summit of either triangle of the pentaedron, we obtain the system 35344 about the pentagonal base. This is a *reversible* hexaedron.

By cutting the first triangle of this with the pentagon and the second with the adjoining quadrilateral, we obtain the system 4364354, which I call an *irreversible* octaedron; it does not read backwards and forwards the same.

The double reversible 3434, by one section of its triangles, becomes 435435, a *double irreversible*, and by another, 436344, a *reversible* heptaedron.

And the system 353535 gives, by cutting the triangles in one way, the *irreversible* 437346345, and, in another way, the *trebly irreversible* 436436436.

These six varieties comprise all the polyedra that have only triedral summits. They are all *irreversible, reversible, doubly irreversible, doubly reversible, trebly irreversible*, or *trebly reversible*.

For if any polyedron exhibited in the faces about the base the fourfold repetition of any period of m faces containing k triangles, the vanishing of the $4k$ triangles would give rise to a fourfold repetition of a period of $m-k$ faces, which would contain k' triangles, k' being not more than k; and this system around the base would reduce, by the vanishing of the $4k'$ triangles in it, to a fourfold repetition of a period of $m-k-k'$ faces; and we should obtain at last a base of $4(m-K)$ sides, admitting no further reduction by the vanishing of triangles, *i. e.* we should obtain a pyramid having a tetraedral summit; which is impossible.

Problem.—An x-edron P being given on an $x-1$-gonal base, and having k triangular faces, it is required to determine how many $(x+k+l)$-edra can be cut from it by the removal of $k+l$ base summits, so that none of the k triangles shall remain untouched, and so that no $(x+k+l)$-edron shall be the reflected image of any other.

First, let P have about its base a series of x faces which read differently both backwards and forwards from every face, *i. e.* let it be an *irreversible* x-edron.

As no two contiguous faces about the base can be triangles, if $x>4$ (for then there would be a tetraedral summit at least, at one extremity of their common side), $k \ngtr \frac{1}{2}(x-1)$, and $k+l \ngtr x-1$.

We are bound to cut each of the k triangles once, which can be done in 2^k ways, giving 2^k different irreversible arrangements of $x+k-1$ faces. Next we have to cut l of the remaining $x-1-k$ angles about the base of P. These may be any l out of $x-1-k$, and this gives us $2^k.\frac{(x-1-k)^{l|-1}}{\lfloor l+1}$ arrangements of our $k+l$ sections. But these will not be all different arrangements. Any one of them will contain e cases of twice cut triangles of P, thus made into e pentagons, and consequently $l-e$ triangles which are not cut from triangles.

Let us suppose $e=2$, the case in which two pentagons are newly made, standing thus,

$$\ldots c353de353f\ldots$$

Of these four triangles, two were introduced in the distribution of our k sections, and two others in that of the l sections, which may happen in 2^2 different ways. This arrangement, $c353de353f$, in which $cdef$ are supposed not to be triangles, will then be found 2^2 times with every disposition of the remaining $k-2$ triangles first cut, and with every $(l-2)$ sections of angles not in triangles, of which angles the number is $x-1-2k$. We have therefore counted the completed arrangement $\ldots c353de353f\ldots$, in the number $2^k\cdot\frac{(x-1-k)^{l|-1}}{\lceil l+1}$,

$$2^2\,.\,2^{k-2}\,.\,\frac{(x-1-2k)^{l-2|-1}}{\lceil l-2+1}\text{ times,}$$

instead of

$$2^{k-2}\,.\,\frac{(x-1-2k)^{l-2|-1}}{\lceil l-2+1}\text{ times.}$$

The same error has been made with every value of $e\not>k$, and $e\not>l$, or $e\not>$ the least of k and l, and this with every set of e twice cut triangles that can be selected out of k. Hence there is an error made in excess, in $2^k\cdot\frac{(x-1-k)^{l|-1}}{\lceil l+1}$, of

$$(2^e-1)\,.\,2^{k-e}\,.\,\frac{(x-1-2k)^{l-e|-1}}{\lceil l-e+1}\cdot\frac{k^{e|-1}}{\lceil e+1},$$

for every value of $e\not>$ the least of k and l; for, in supposing e twice cut triangles, we assume that $e\not>k$, and that $k+l-k\not<e$. The number required in the problem is thus proved to be

$$2^k\cdot\frac{(x-k-1)^{l|-1}}{\lceil l+1}-\Sigma_e\cdot(2^e-1)2^{k-e}\cdot\frac{(x-1-2k)^{l-e|-1}}{l+1-e}\cdot\frac{k^{e|-1}}{\lceil e+1},$$

for all positive values of e not greater than either k or l: which was to be found.

This function I shall denote by the symbol $ii.(x,k,l)$. It expresses the number of $(x+k+l)$-edra that can be made from any x-edron having an $(x-1)$-gonal base and k triangular faces, of which no two are contiguous, by removing $k+l$ of the summits about the base, so that no edge shall be entirely removed, and that no one of the k triangles shall remain untouched. Of course $k+l\not>x-1$. Its values are

$$ii(x,k,0)=2^k;\quad ii(x,k,1)=2^{k-1}(2x-3k-2);$$

$$ii(x,1,l)=2\cdot\frac{(x-2)^{l|-1}}{\lceil l+1}-\frac{(x-3)^{l-1|-1}}{\lceil l};$$

$$ii(x,k,2)=2^{k-1}.\left[(x-3)(x-4)-k.(x-1-2k)\right]-3.2^{k-3}.k.(k-1);$$

$$ii(x,2,l)=4\cdot\frac{(x-3)^{l|-1}}{\lceil 1+l}-4\cdot\frac{(x-5)^{l-1|-1}}{\lceil l}-3\cdot\frac{(x-5)^{l-2|-1}}{\lceil l-1};$$

&c. &c.

Let $\mathrm{I}(x,k)$ be the total number of irreversible x-edra on an $\overline{x-1}$-gonal base that have k triangular faces. Then

$$\mathrm{I}(x,k)\{ii.(x,k,l)\},\text{ part of }\mathrm{I}(x+k+l,\,k+l),$$

is the whole number of $(x+k+l)$-edra that can be cut, from irreversible x-edra having k triangles, so as to have $k+l$ triangles. Others can be cut to be also part of

$I(x+k+l,\ k+l)$, from other x-edra having k' triangles, by removing $k'+l'$ summits about the base, none of the k' triangles being untouched, if $k'+l'=k+l$.

Next let our subject of operation be an irreversible containing more than one period of faces about the base. If it be a doubly irreversible, the base will be $2x$-gonal, as every face will be opposite to a similar one. We have then two periods each of x faces, and if we operate on one of these so as to remove angles, leaving no triangle in the period untouched, and then repeat the operation exactly in order in the other period, we shall obtain a doubly irreversible for our result.

Let $I^2(2x+1,\ 2k)$ be the whole number of doubly irreversible $(2x+1)$-edra on a $2x$-gonal base, having $2k$ triangles. Then $ii.\left(x+1,\ k,\ \frac{l}{2}\right)$ is the number of ways in which we can remove $k+\frac{l}{2}$ summits from one period, and therefore

$$I^2(2x+1,\ 2k)ii\left(x+1,\ k,\ \frac{l}{2}\right),\ \text{part of } I^2(2x+1+2k+l,\ 2k+l),$$

is the whole number of doubly irreversibles having $2k+l$ triangles, and a $(2x+2k+l)$-gonal base, that can be cut from all the $I^2(2x+1,\ 2k)$ polyedra before us.

If l is not an even number, $ii\left(x+1,\ k,\ \frac{l}{2}\right)$ is to be considered nothing; for it is impossible to remove a fractional number of summits.

We can cut also from these $I^2.(2x+1, 2k)$ doubly irreversibles a number of singly irreversibles. If each of these doubly irreversibles were single, it would give rise to $ii(2x+1, 2k,\ l)$ singly irreversibles; but the double character of the subject of operation causes every method of removing $2k+l$ angles, which is not alike in both periods of the subject, to appear twice in the number just written. That is, we are to subtract from this number all the doubly irreversibles that can be made, and take half the remainder, which is, after division,

$$\tfrac{1}{2}\left\{ii(2x+1,\ 2k,\ l)-ii\left(x+1,\ k,\ \frac{l}{2}\right)\right\}.$$

The second term of this is zero when l is odd. We obtain thus for the number of singly irreversibles that can be cut to have $2k+l$ triangles from all the $I^2(2x+1,\ 2k)$ under consideration,

$$I^2.(2x+1,\ 2k).\tfrac{1}{2}\left\{ii(2x+1,\ 2k,\ l)-ii\left(x+1,\ k,\ \frac{l}{2}\right)\right\},$$

a part of $I(2x+1+2k+l,\ 2k+l)$.

Next let us consider the operations that can be effected on triply irreversible $(3x+1)$-edra having $3k$ triangles. Let their number be $I^3(3x+1,\ 3k)$. It is easily proved by a repetition of the preceding argument, that $ii\left(x+1,\ k,\ \frac{l}{3}\right)$ is the number of triply irreversibles that can be cut from each of them, and that

$$\tfrac{1}{3}\left\{ii(3x+1,\ 3k,\ l)-ii\left(x+1,\ k,\ \frac{l}{3}\right)\right\}$$

is that of the singly irreversibles, where the appearance of a fraction in the function ii, reduces it, as it always must, to zero. That is, we obtain

$$I^3(3x+1, 3k).ii\left(x+1, k, \frac{l}{3}\right), \quad \text{as } I^3.(3x+3k+l+1, 3k+l),$$

and $$I^3(3x+1, 3k)\tfrac{1}{3}\left\{ii(3x+1, 3k, l)-ii\left(x+1, k, \frac{l}{3}\right)\right\}$$

as a portion of $$I(3x+3k+l+1, 3k+l).$$

From an irreversible no reversible can be cut by this removing of summits; for as the summits of triangles of a reversible correspond in pairs, the arrangement of faces about the base will still be reversible, if all the triangles are supposed to vanish.

Let us now operate on a reversible polyedron, whose faces about the base read backwards and forwards alike.

There will be a certain period *abc....klm* reversed, in one of the three ways,

abc....klmlk....cb,

abc....klmmlk....cb,

or *abc....klmmlk....cba*.

There is in any of these what may be called an axis of reversion, which in the first passes through the faces *a* and *m*, in the second through *a* and between two *m*'s, in the third between two *a*'s and between two *m*'s.

It is evident that the number of triangles about the base of a reversible cannot be odd, unless the axis of reversion passes through a triangle; as all faces recur in order reversed, through which that axis does not pass; and the base must be $(2x+1)$-gonal, if the axis passes through one face only. The third case, of an axis of reversion passing through no face, does not occur when all the summits are triedral.

First let the base be even, and let the number of triangles be even also; we have to consider the operations practicable upon a $(2x+1)$-edron R reversible, with $2k$ triangles. Some $(2x+1+2k+l)$-edra can be cut from it reversible, and some irreversible, by the removing of $2k+l$ summits of the base, leaving none of the $2k$ triangles untouched.

A reversible so cut will have on either side of its axis of reversion half the $2k+l$ added triangles, unless it passes through an added triangle, in which case it will have on either side $\frac{1}{2}(2k+l-1)$ of them.

Let $l=2l'$; then the number of possible operations on one side of the axis of reversion, which exhibits x summits, is $ii(x, k, l')$; each of which gives by repeating it backwards one of our reversible $(2x+1+2k+2l')$-edra. As the axis does not pass through a summit, l cannot be odd, for the added triangles are all in pairs.

If then $R(2x+1, 2k)$ be the total number of reversible $(2x+1)$-edra having $2k$ triangles, we obtain

$$R(2x+1, 2k).ii\left(x, k, \frac{l}{2}\right), \quad \text{part of } R(2x+1+2k+l, 2k+l).$$

Now take the reversible $(2x+1)$-edron R′ having $2k+1$ triangles; the axis of reversion passes through one of them. As this triangle cannot be untouched, it must be twice touched, so that we have only $(2k+1+l-2)$ other sections to make, one half of these on the $x-1$ summits on one side of the axis, neglecting the summit of the central triangle. The possible operations are

$$ii\left(x, k, \frac{l-2}{2}\right);$$

or, if $R(2x+1, 2k+1)$ be the total number of reversible $(2x+1)$-edra having $2k+1$ triangles, we obtain

$$R(2x+1, 2k+1)ii\left(x, k, \frac{l-2}{2}\right), \text{ part of } R(2x+2+2k+l, 2k+1+l).$$

We take now a reversible R″ having a $(2x-1)$-gonal base, and $2k$ triangles.

The axis of reversion passes through a summit and through a face, which we shall suppose to be not a triangle. It is not difficult to prove that it can be no triangle if $2x-1>3$, all the summits being triedral.

We have on either side of the axis $\frac{1}{2}(2x-2)$ summits, that in the axis being neglected, and half the $2k$ triangles. We have to distribute $k+\frac{l}{2}$ sections on these $x-1$ summits. The number of ways to do it is

$$ii\left(x, k, \frac{l}{2}\right),$$

which requires l to be even; or if l be odd, we may cut the summit in the axis, and distribute $k+\frac{l-1}{2}$ sections on the $x-1$ summits, giving $ii\left(x, k, \frac{l-1}{2}\right)$. These operations reversed on the other side of the axis give us all the possible results.

If then $R(2x, 2k)$ be the total number of reversible $2x$-edra having $2k$ triangles, we can cut from all these

$$R(2x, 2k)\left\{ii\left(x, k, \frac{l}{2}\right)+ii\left(x, k, \frac{l-1}{2}\right)\right\}$$

$(2x+2k+l)$-edra reversible, having $2k+l$ triangles. One of the terms in the second factor is always zero. The polyedra so cut are a portion of $R(2x+2k+l, 2k+l)$.

When $k+l=x-1$, all the angles about the base of the reversible with k triangles are cut, and the result is of necessity reversible.

But if $k+l<x-1$, some of the results of $k+l$ sections will be irreversible. The whole number of results, if we treated the n-edron R or R′ or R″ as irreversible, would be $ii(n, k, l)$, but these are not all different.

They will all, except the reversible ones, have a different order on the two sides of the axis of reversion; and each irreversible will occur twice, the second time reverted by exchanging the arrangements of the two sides of the axis, so as to make a polyedron and its reflected image. As we are not to count these reflexions, we have to subtract from $ii(n, k, l)$ all the possible reversible results, and divide the remainder by two. That is, by what precedes, we obtain from R, R$'$ and R$''$

$$\tfrac{1}{2}\left\{ii(2x+1, 2k, l)-ii\left(x, k, \frac{l}{2}\right)\right\},$$

$$\tfrac{1}{2}\left\{ii(2x+1, 2k+1, l)-ii\left(x, k, \frac{l-2}{2}\right)\right\},$$

$$\tfrac{1}{2}\left\{ii(2x, 2k, l)-ii\left(x, k, \frac{l}{2}\right)-ii\left(x, k, \frac{l-1}{2}\right)\right\}.$$

It is to be understood in these formulæ that

$2k+l<2x$ in the first,

$2k+1+l<2x$ in the second,

$2k+l<2x-1$ in the third.

If we multiply the first by $R(2x+1, 2k)$, the second by $R(2x+1, 2k+1)$, and the third by $R(2x, 2k)$, we obtain the corresponding portions of $I(2x+1+2k+l, 2k+l)$, $I(2x+2+2k+l, 2k+1+l)$, and $I(2x+2k+l, 2k+l)$.

Let us next operate on a doubly reversible $(4x+1)$-edron, with $4k$ triangles. All these are cut from the pentaedron 3434, and by the addition of an even number of triangles in each period. The operations by which doubly reversibles are cut from a doubly reversible are simply those whereby reversibles are cut from the reversible period of $2x$ summits, containing $2k$ triangles, being in number $ii\left(x+1, k, \frac{l}{4}\right)$ by what has preceded. Or, if $R^2(4x+1, 4k)$ be the number of $(4x+1)$-edra doubly reversible, with $4k$ triangles,

$$R^2(4x+1, 4k).ii.\left(x+1, k, \frac{l}{4}\right)=R^2(4(x+k)+l+1, 4k+l),$$

there being no more; for one of these latter can be cut from nothing but a doubly reversible with $4k+l$ faces fewer.

From the same $(4x+1)$-edra can be cut doubly irreversibles, namely so many as the irreversibles producible from sections of one period, or from a reversible $(2x+1)$-edron with $2k$ triangles. This number is, as just proved,

$$\tfrac{1}{2}\left\{ii\left(2x+1, 2k, \frac{l}{2}\right)-ii\left(x+1, k, \frac{l}{4}\right)\right\},$$

which multiplied by $R^2.(4x+1, 4k)$, constitutes a portion of $I^2.(4x+4k+l+1, 4k+l)$.

As a doubly reversible is also a reversible, reversibles can be cut from it. It is to be observed that, as a reversible, it has two axes of reversion, as indeed every $2m$-ly reversible has. Thus the enneaedron 35363536 has an axis through the two pentagons and another through the hexagons. If we operate on one side of the axis for irreversible results, and revert our operations on the other side, we obtain reversibles. The number of such results in either position of the axis of reversion is that of the irreversibles producible by $\left(2k+\frac{l}{2}\right)$ sections of a reversible $(2x+1)$-gon with $2k$

triangles, or

$$\tfrac{1}{2}\left\{ii\left(2x+1,\ 2k,\ \frac{l}{2}\right)-ii\left(x+1,\ k,\ \frac{l}{4}\right)\right\};$$

whence we obtain, from both positions of the axis,

$$R^2.(4x+1,\ 4k).\left\{ii\left(2x+1,\ 2k,\ \frac{l}{2}\right)-ii\left(x+1,\ k,\ \frac{l}{4}\right)\right\}$$

as part of $\quad R(4x+1+4k+l,\ 4k+l).$

If we forget for a moment the character of one of these $R^2(4m+1,\ 4k)$, and treat it as an irreversible, we obtain by $4k+l$ sections $ii(4m+1,\ 4k,\ l)$ results. Of these the doubly reversibles can occur only once, every doubly irreversible will occur twice, in one result as the reflexion of the other; every reversible will occur twice, the operations in the first period in the second result being those of the second period in the first; and every irreversible will occur four times, twice by the exchange of the operations on the first period for those on the second, and twice again by the reversion of all the operations, producing reflected images of two preceding results.

That is, if we subtract from $ii(4x+1,\ 4k,\ l)$ all the doubly reversibles, twice the reversibles, and twice the doubly reversibles that can be cut from a doubly reversible $(4x+1)$-gon having $4k$ triangles, by $4k+l$ sections, there remains four times the number of irreversibles that can be cut from it, by $4k+l$ sections.

This remainder, divided by 4, is

$$\begin{aligned}&\tfrac{1}{4}\Bigg[ii(4x+1,\ 4k,\ l)-ii\left(x+1,\ k,\ \frac{l}{4}\right)\\ &\qquad-2\left\{ii\left(2x+1,\ 2k,\ \frac{l}{2}\right)-ii\left(x+1,\ k,\ \frac{l}{4}\right)\right\}\\ &\qquad-\left\{ii\left(2x+1,\ 2k,\ \frac{l}{2}\right)-ii\left(x+1,\ k,\ \frac{l}{4}\right)\right\}\Bigg]\\ &=\tfrac{1}{4}\left[ii(4x+1,\ 4k,\ l)+2ii\left(x+1,\ k,\ \frac{l}{4}\right)-3ii\left(2x+1,\ 2k,\ \frac{l}{2}\right)\right],\end{aligned}$$

which, multiplied by $R^2(4x+1,\ 4k)$, is to be added to $I(4x+1+4k+l,\ 4k+l)$.

It remains that we handle now trebly reversible $(6x+1)$-edra, having $6k$ triangles.

If $x>1$, the number of triangles in a triply reversible cannot be less than $6k$, as they are all cut from the heptaedron 535353, by an even number of sections in every period. By operating on one reversible period of $2x$ summits and $2k$ triangles for reversibles by $\frac{1}{3}(6k+3l)$ sections, we obtain all the triply reversibles. The number of these so found is $ii.\left(x+1,\ k,\ \frac{l}{2}\right)$, giving so many $(6x+1+6k+3l)$-edra triply reversible with $6k+3l$ triangles, for each subject; and in all

$$R^3(6x+1,\ 6k)\left(ii\left(x+1,\ k,\ \frac{l}{2}\right)\right)=R^3(6k+1+6k+3l,\ 6k+3l).$$

The number of triply irreversibles is equal to that of the irreversibles producible from a reversible period of $2x$ summits and $2k$ triangles by $\frac{1}{3}(6k+3l)$ sections, and is, by what precedes,

$$\tfrac{1}{2}\left\{ii(2x+1,\, 2k,\, l)-ii\left(x+1,\, k,\, \frac{l}{2}\right)\right\},$$

from each subject of operation. This, multiplied by $R^3(6x+1, 6k)$, is to be added to $I^3(6x+1+6k+3l,\, 6k+3l)$.

The reversibles obtainable from a triply irreversible are found only about one axis of reversion. Thus 735373537353 has only one axis through a heptagon and a pentagon. We are to cut by $\frac{1}{2}(6k+2l)$ sections on one side of this axis all possible irreversibles from a reversible system of $3x$ summits and $3k$ triangles. These results reverted on the other side of the axis, will give all possible reversibles. Among these will be all triply reversible $(6x+1+6k+2l)$-edra, with $6k+2l$ triangles, for these are all reversible; and none of these can occur more than once. We have these to subtract from our results, leaving

$$ii(3x+1,\, 3k,\, l)-ii\left(x+1,\, k,\, \frac{l}{3}\right)$$

reversibles from every subject; which number, multiplied by $R^3(6x+1,\, 6k)$, forms part of $R(6x+1+6k+2l,\, 6k+2l)$.

If we treated a triply reversible as an irreversible by $6k+l$ sections, we should obtain

$$ii(6x+1,\, 6k,\, l) \text{ results.}$$

Among these every triply reversible is found once; every triply irreversible twice, one place showing the reflected image of the other; every reversible three times, each time the same operations commencing at a different period; and every irreversible six times, being begun both backwards and forwards in three different periods. If then we subtract from $ii(6x+1,\, 6k,\, l)$ every triply reversible, twice the triply irrevisibles, and thrice the reversibles that can be made by $6k+l$ sections of a triply reversible $(6x+1)$-edron having $6k$ triangles, there will remain six times the number of irreversibles that can be so cut from the same. This remainder, after division by 6, is

$$\tfrac{1}{6}\Big\{ii(6x+1,\, 6k,\, l)-ii\left(x+1,\, k,\, \frac{l}{6}\right)-ii\left(2x+1,\, 2k,\, \frac{l}{3}\right)+ii\left(x+1,\, k,\, \frac{l}{6}\right)$$

$$-3ii\left(3x+1,\, 3k,\, \frac{l}{2}\right)+3ii\left(x+1,\, k,\, \frac{l}{6}\right)\Big\}$$

$$=\tfrac{1}{6}\left\{ii(6x+1,\, 6k,\, l)+3ii\left(x+1,\, k,\, \frac{l}{6}\right)-ii\left(2x+1,\, 2k,\, \frac{l}{3}\right)-3ii\left(3x+1,\, 3k,\, \frac{l}{2}\right)\right\},$$

which, multiplied by $R^3(6x+1,\, 6k)$, forms part of $I(6x+1+6k+l,\, 6k+l)$.

It is most convenient to treat the case of the triply reversible having only three triangles by itself. From this heptaedron can be cut one triply reversible by cutting every summit of the base. One triply irreversible only can be made, a decaedron, by cutting each triangle once. Two reversibles can be cut, by four sections, differing in the manner of cutting the triangle through which the axis of reversion does not pass, giving two reversible 11-edra. Three irreversibles can be cut from it, one by three, another by four, a third by five sections, as is evident on a moment's consideration, giving a 10-edron, 11-edron, and 12-edron.

We can now collect into one group all the formulæ above deduced, which contain the complete solution of our problem; to find the number of these x-edra on an $(x-1)$-gonal base. There is no ambiguity in the case of two $(x-1)$-gonal faces, for the figure is always identical with itself whichever be considered the base, and can have only two triangles.

Let $\mathrm{II}(x, k, l)$ or $\mathrm{IR}(x, k, l)$ be the number of irreversible $(x+k+l)$-edra that can be cut to have $(k+l)$ triangles from an irreversible or reversible x-edron having k triangles, the capital on the right denoting the subject of operation: $(k+l)<x$.

$$\mathrm{II}(x, k, l) = ii(x, k, l),$$

$$\mathrm{II}^2(2x+1, 2k, l) = \tfrac{1}{2}\left\{ii(2x+1, 2k, l)-ii\left(x+1, k, \frac{l}{2}\right)\right\},$$

$$\mathrm{II}^3(3x+1, 3k, l) = \tfrac{1}{3}\left\{ii(3x+1, 3k, l)-ii\left(x+1, k, \frac{l}{3}\right)\right\},$$

$$\mathrm{I}^2\mathrm{I}^2(2x+1, 2k, l) = ii\left(x+1, k, \frac{l}{2}\right),$$

$$\mathrm{RR}(2x+1, 2k, l) = ii\left(x+1, k, \frac{l}{2}\right),$$

$$\mathrm{RR}(2x+1, 2k+1, l) = ii\left(x, k, \frac{l-2}{2}\right),$$

$$\mathrm{RR}(2x, 2k, l) = ii\left(x, k, \frac{l}{2}\right)+ii\left(x, k, \frac{l-1}{2}\right),$$

$$\mathrm{IR}(2x+1, 2k, l) = \tfrac{1}{2}\left\{ii(2x+1, 2k, l)-ii\left(x+1, k, \frac{l}{2}\right)\right\},$$

$$\mathrm{IR}(2x+1, 2k+1, l) = \tfrac{1}{2}\left\{ii(2x+1, 2k+1, l)-ii\left(x, k, \frac{l-2}{2}\right)\right\},$$

$$\mathrm{IR}(2x, 2k, l) = \tfrac{1}{2}\left\{ii(2x, 2k, l)-ii\left(x, k, \frac{l}{2}\right)-ii\left(x, k, \frac{l-1}{2}\right)\right\};$$

$$\mathrm{R}^2\mathrm{R}^2(4x+1, 4k, l) = ii(x+1, k, \tfrac{1}{4}l),$$

$$\mathrm{I}^2\mathrm{R}^2(4x+1, 4k, l) = \tfrac{1}{2}\{ii(2x+1, 2k, \tfrac{1}{2}l)-ii(x+1, k, \tfrac{1}{4}l)\},$$

$$\mathrm{RR}^2(4x+1, 4k, l) = ii(2x+1, 2k, \tfrac{1}{2}l)-ii(x+1, k, \tfrac{1}{4}l),$$

$$\mathrm{IR}^2(4x+1, 4k, l) = \tfrac{1}{4}\left[ii(4x+1, 4k, l)+2ii\left(x+1, k, \frac{l}{4}\right)-3ii(2x+1, 2k, \tfrac{1}{2}l)\right];$$

$$\mathrm{R}^3\mathrm{R}^3(6x+1, 6k, l) = ii(x+1, k, \tfrac{1}{6}l), \quad \mathrm{R}^3\mathrm{R}^3(7, 3, 3)=1,$$

$$\mathrm{I}^3\mathrm{R}^3(6x+1, 6k, l) = \tfrac{1}{2}\{ii(2x+1, 2k, \tfrac{1}{3}l)-ii(x+1, k, \tfrac{1}{6}l)\},$$

$$\mathrm{RR}^3(6x+1, 6k, l) = ii(3x+1, 3k, \tfrac{1}{2}l) - ii(x+1, k, \tfrac{1}{6}l), \quad \mathrm{RR}^3(7, 3, 1)=2,$$

$$\mathrm{IR}^3(6x+1, 6k, l) = \tfrac{1}{6}\{ii(6x+1, 6k, l)+3ii(x+1, k, \tfrac{1}{6}l) - ii(2x+1, 2k, \tfrac{1}{3}l) - 3ii(3x+1, 3k, \tfrac{1}{2}l)\},$$

$$\mathrm{IR}^3(7, 3, 2) = \mathrm{IR}^3(7, 3, 1) = \mathrm{IR}^3(7, 3, 0) = 1:$$

$$\mathrm{I}^n\mathrm{R}^m(x+1, k, x-k)=0,$$

for if all the summits of a reversible are cut, the result is reversible.

In $\mathrm{I}^n\mathrm{I}^m(x, k, l)$, $\mathrm{I}^n\mathrm{R}^m(x, k, l)$, $\mathrm{R}^n\mathrm{R}^m(x, k, l)$, the second capital marks the character and multiplicity of the x-edra having k triangles, from which are cut the $(x+k+l)$-edra having $k+l$ triangles, of which the character and multiplicity are denoted by the first.

To show the use of these equations, we can easily by trial verify the following:—

$$\begin{array}{lll} \mathrm{R}^3(4, 3)=1, & \mathrm{R}^2(5, 2)=1, & \mathrm{R}(6, 2)=1, \\ \mathrm{R}(7, 2)=1, & \mathrm{R}^3(7, 3)=1, & \mathrm{I}^2(7, 2)=1, \\ \mathrm{R}(8, 2)=2, & \mathrm{I}(8, 2)=1, & \mathrm{I}(8, 3)=1, \\ \mathrm{R}(9, 3)=2, & \mathrm{R}(9, 2)=2, & \mathrm{R}^2(9, 4)=1, \\ \mathrm{I}(9, 3)=3, & \mathrm{I}(9, 2)=2, & \mathrm{I}^2(9, 2)=2; \end{array}$$

then to find the decaedra on a 9-gonal base, with only triedral summits, we first write down the classes,

$$\mathrm{R}(10, 2)=\mathrm{R}(8, 2).\mathrm{RR}(8, 2, 0);$$

$$\mathrm{I}(10, 2)=\mathrm{R}(8, 2).\mathrm{IR}(8, 2, 0)+\mathrm{I}(8, 2).\mathrm{II}(8, 2, 0);$$

$$\mathrm{I}(10, 3)=\mathrm{R}^3(7, 3).\mathrm{IR}^3(7, 3, 0)+\mathrm{R}(7, 2).\mathrm{IR}(7, 2, 1)+\mathrm{I}^2(7, 2).\mathrm{II}^2(7, 2, 1);$$

$$\mathrm{I}^3(10, 3)=\mathrm{R}^3(7, 3).\mathrm{I}^3\mathrm{R}^3(7, 3, 0);$$

$$\mathrm{R}(10, 4)=\mathrm{R}(6, 2).\mathrm{RR}(6, 2, 2);$$

$$\mathrm{I}(10, 4)=\mathrm{R}(6, 2).\mathrm{IR}(6, 2, 2).$$

That is, by what precedes,

$$\mathrm{R}(10, 2)=2.ii(4, 1, 0)=2.2=4;$$

$$\mathrm{I}(10, 2)=2.\tfrac{1}{2}\{ii(8, 2, 0)-ii(4, 1, 0)\} = 2.\tfrac{1}{2}\{4-2\}=2;$$

$$\mathrm{I}(10, 3)=1.\tfrac{1}{6}\{ii(7, 3, 0)-ii(3, 1, 0)\}+1.\tfrac{1}{2}ii(7, 2, 1)+1.\tfrac{1}{2}ii(7, 2, 1) = \tfrac{1}{6}\{8-2\}+\tfrac{12}{2}+\tfrac{12}{2}=13;$$

$$\mathrm{I}^3(10, 3)=1.1=1;$$

$$\mathrm{R}(10, 4)=1.ii(3, 1, 1)=1.(6-5)=1;$$

$$\mathrm{I}(10, 4)=1.\tfrac{1}{2}\{ii(6, 2, 2)-ii(3, 1, 1)\}=1.\tfrac{1}{2}\{72-108+41-1\}=2.$$

Next to find the hendecaedra, we write down

$I(11, 5) = R(6, 2).IR(6, 2, 3) = 0;$

$R(11, 5) = R(6, 2).RR(6, 2, 3);$

$R(11, 4) = R(7, 2).RR(7, 2, 2) + R^3(7, 3).RR^3(7, 3, 1);$

$I(11, 4) = R(7, 2).IR(7, 2, 2) + I^2(7, 2).II^2(7, 2, 2) + R^3(7, 3).IR^3(7, 3, 1);$

$I^2(11, 4) = I^2(7, 2).I^2I^2(7, 2, 2);$

$R(11, 3) = R(8, 2).RR(8, 2, 1);$

$I(11, 3) = R(8, 2).IR(8, 2, 1) + I(8, 2).II(8, 2, 1) + I(8, 3).II(8, 3, 0);$

$R(11, 2) = R(9, 2).RR(9, 2, 0);$

$I(11, 2) = R(9, 2).IR(9, 2, 0) + I(9, 2).II(9, 2, 0) + I^2(9, 2).II^2(9, 2, 0);$

$I^2(11, 2) = I^2(9, 2).I^2I^2(9, 2, 0).$

That is—

$R(11, 5) = 1.ii(3, 1, 1) \qquad = 1;$

$R(11, 4) = 1.ii(4, 1, 1) + 1.2 \qquad = 5;$

$I(11, 4) = 1.\tfrac{1}{2}\{ii(7, 2, 2) - ii(4, 1, 1)\}$

$\qquad + 1.\tfrac{1}{2}\{ii(7, 2, 2) - ii(4, 1, 1)\} + 1.1 \qquad = 11;$

$I^2(11, 4) = 1.ii(4, 1, 1) \qquad = 3;$

$R(11, 3) = 2.ii(4, 1, 0) \qquad = 4;$

$I(11, 3) = 2.\tfrac{1}{2}\{ii(8, 2, 1) - ii(4, 1, 0)\}$

$\qquad + 1.ii(8, 2, 1) + 1.ii(8, 3, 0)$

$\qquad = 2.7 + 1.16 + 1.8 \qquad = 38;$

$R(11, 2) = 2.ii(5, 1, 0) = 2.2 \qquad = 4;$

$I(11, 2) = 2.\tfrac{1}{2}\{ii(9, 2, 0) - ii(5, 1, 0)\} + 2.ii(9, 2, 0)$

$\qquad + 2.\tfrac{1}{2}\{ii(9, 2, 0) - ii(5, 1, 0)\} = 2.1 + 2.4 + 2.1 = 12;$

$I^2(11, 2) = 2.ii(5, 1, 0) \qquad = 4.$

As a verification, it may be worth while to write down these eighty-two 11-edra, to show the faces in order about the 10-gonal base.

R(11, 5) is 3537353636;

R(11, 4) are 5383535453, 4383453635, 3464363636, 4373537345, 3463536437;

I(11, 4) are 4438353635, 6346436363, 5437436363, 3543835354, 3637374354, 6353653463, 6353644373, 6346354373, 5353736345, 6353644373, 4437363536;

I^2(11, 4) are 5353653536, 6346363463, 5437354373;

R(11, 3) are 4439344535, 5347435535, 4355534636, 3445443737;

I(11, 3) are 5444439353, 5734446353, 6634455353, 7534545353, 8435445353, 5463447353, 5553456353, 5643546353, 5436444383, 8436354443, 5436634463, 6436356344, 5436453473, 7346354534, 5436543563, 6436355435, 5439344453, 5438435453, 5437443653, 5437535453, 5436444383, 5436383444, 5436354435, 5436374354, 5443744373, 5534653463, 6443743643, 5443743734, 5443834634, 6443834543, 6534643643, 5534734634, 5534643734, 6534734543, 5443753463, 5443843553, 5534653463, 5534644373;

R(11, 2) are 4534843544, 44431034444, 5435653454, 6344644364;

I(11, 2) are 4534934444, 6344653454, 6734445443, 6643545443, 5734445534, 5643545534, 5834444543, 4834444634, 5743544543, 4743544634, 6444374534, 6553455443;

$I^2(11, 2)$ are 5553455534, 7444374443, 6544365443, 6453464534.

The dodecaedra are found by rather less calculation than the hendecaedra, forming only eight classes, as follows:—

$I(12,5) = R^2(7,3).IR^3(7,3,2)+I^2(7,2).II^2(7,2,3)+R(7,2).IR(7,2,3)$,

$R(12,5)=R(7,2)RR(7,2,3)+R^3(7,3).RR^3(7,3,2)$,

$I(12,4) = I(8,3)II(8,3,1)+I(8,2).II(8,2,2)+R(8,2).IR(8,2,2)$,

$R(12,4)=R(8,2).RR(8,2,2)$,

$I(12,3) = I(9,3).II(9,3,0)+I(9,2)II(9,2,1)+R(9,3).IR(9,3,0)+R(9,2).IR(9,2,1)$,

$R(12,3)=R(9,3).RR(9,3,0)+R(9,2)RR(9,2,1)$,

$I(12,2) = I(10,2)II(10,2,0)+R(10,2).IR(10,2,0)$,

$R(12,2)=R(10,2).RR(10,2,0)$.

These are, by what precedes, all given numbers; and, by continuing the process, we can finally obtain all the x-edra on an $(x-1)$-gonal base, numbered in their proper classes, which have only triedral summits.

I have generalized the expressions $I^nI^m(x,k,l)$, $R^nR^m(x,k,l)$, $I^nR^m(x,k,l)$, which the theory requires for enumerating all the x-edra having an $(x-1)$-gonal base, and any summits whatever; but the formulæ are not worth producing. The number of distinctions to be made is too great to be of any ready use. If the x-edra having an $(x-1)$-gonal base were classed and enumerated according to their summits, it would be possible to count all the $(x+h)$-edra on the same $(x-1)$-gonal base, by removing summits not in the base, thus producing *crown-faces*, and by the vanishing of edges about the crowns and base, thus producing faces contiguous, but not collateral. That is, it would be possible to enumerate and classify the N-edra.